A Study on Rings of Continuous Functions

A Thesis

Presented to

the Faculty of the Department of Mathematics

Appalachian State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Keith David Huffstetler

August 1971

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With every topological space there may be associated two algebraic structures, namely, the ring of real-valued continuous bounded functions and the ring of all real-valued continuous functions defined on that space. The present paper is concerned with the study of these function rings. Some fundamental properties of these function rings are established, and a foundation is laid for further study and applications to topological questions.

Chapter one contains the preliminaries necessary for a study of rings of continuous functions, i. e., basic terminology and notation used. In chapter two, the ideas of zero-sets and completely separated sets are introduced and Urysohn's Extension Theorem, the basic result about C*-embedding, is presented. Included in chapter three is a brief look at some relationships between the components of a topological space X and algebraic properties of C(X), in particular, the square roots of unity elements. Arguements are presented in chapter four concerning the restriction of the topological spaces to be considered to completely regular spaces, and some basic results obtained by doing so are included. It is noted that several additional results are obtained by considering normal spaces, and some of these results are included in chapter five.

INTRODUCTION

With every topological space there may be associated two algebraic structures, namely, the ring of real-valued continuous bounded functions and the ring of all real-valued continuous functions defined on that space. In most cases, these rings have a great number of interesting algebraic properties, which are connected with corresponding topological properties of the space on which they are defined.

The present paper is concerned with the study of these function rings, with a view toward establishing their fundamental properties and laying the foundation for applications to purely topological questions. Definitions or explanations of terminology and notation are included concerning a number of ideas, but for those left undefined the reader is referred to T. O. Moore's <u>Elementary General Topology</u>, from which most of the topological notation was adopted.

The first chapter of this paper contains the preliminaries necessary for a study of rings of continuous functions, i. e., basic terminology and notation used. In chapter two, the ideas of zero-sets and completely separated sets are introduced and Urysohn's Extension Theorem, the basic result about C*-embedding, is presented. Chapter three contains a look at connected spaces and at components of spaces and how they relate to their function rings. Arguements are presented in chapter four concerning the restriction of the topological spaces to be considered to completely regular spaces, and some basic results obtained by doing so are included. It is noted that several additional results are obtained by considering normal spaces, and some of these results are included in chapter five.

The majority of the results included in this paper were previously known. However, the majority of the proofs are those of the author.

TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

The set of all functions from a space X into the real numbers R will be denoted by R^X. C(X), sometimes shortened to C, will denote the set of all continuous real-valued functions defined on the topological space X; and C*(X), sometimes shortened to C*, will denote the set of all bounded continuous real-valued functions defined on a space X.

For f and g elements of R^X, define addition and multiplication by $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for all x in X. That the two operations thus defined are associative and commutative, and that the distributive property holds, follows from the corresponding statements concerning the field R. The zero element is the constant function 0 and the unity element is the constant function 1. The additive inverse -f of f is characterized by $(-f)(x) = -f(x)$. We thus have that R^X is a commutative ring with unity element (provided that X is not empty). If it exists, the multiplicative inverse f^{-1} of f is characterized by $f^{-1}(x) = \frac{1}{f(x)}$ and will be denoted by $\frac{1}{f}$ to avoid confusion with the inverse image of the function f.

Now, since the sum of two continuous functions is continuous, the product of two continuous functions is continuous, if $f \in C(X)$ then $-f \in C(X)$, and 0 and 1 are continuous, we have that $C(X)$ is a subring of R^X with unity element. Similarly, $C^*(X)$ is closed under the same operations and hence is a subring of $C(X)$ with unity element. Thus, for the remainder of this paper, all rings considered are assumed to have a unity element.

The partial ordering on R^X, and hence $C(X)$ and $C^*(X)$, is defined by $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in X.

In a partially ordered set, the symbol $a \vee b$ denotes $\sup\{a, b\}$ and $a \wedge b$ denotes $\inf\{a, b\}$, and when both $a \vee b$ and $a \wedge b$ exist for all a, b in A, A is called a lattice. A subset S is a sublattice of A provided that, for all x, y in S, the elements $x \vee y$ and $x \wedge y$ of A belong to S. A mapping h from a lattice A into a lattice B is a lattice homomorphism into B provided that $h(a \vee b) = h(a) \vee h(b)$ and $h(a \wedge b) = h(a) \wedge h(b)$.

For functions f and g defined on X, $f \vee g$ is defined by $(f \vee g)(x) = f(x) \vee g(x)$, and $f \wedge g$ is defined by $(f \wedge g)(x) = f(x) \wedge g(x)$ for all x in X. It is easy to see that if f is continuous then $|f|$ defined by $|f|(x) = |f(x)|$ is continuous. Also, since $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$, for f and g continuous, $f \vee g$ and $f \wedge g$ are continuous, and $C(X)$ is a sublattice of R^X, and $C^*(X)$ is a sublattice of $C(X)$.

CHAPTER II

ZERO-SETS AND COMPLETELY SEPARATED SETS

In studying relationships between topological properties of a space X and algebraic properties of $C(X)$, it is natural to look at subsets of X of the form $f^{-1}(r) = \{x \in X : f(x) = r\}$ where $f \in C(X)$ and $r \in R$. Clearly, these sets are closed. Of particular interest are the sets of this form where $r = 0$.

Definition 2.1: For $f \in C(X)$, $\{x \in X : f(x) = 0\}$ is called the zero-set of f and is denoted by $Z(f)$. The family of all zero-sets of a space X is denoted by $Z(X)$.

Noting that f^n is defined by $f^n(x) = (f(x))^n$ for any positive integer n, we have that $Z(f) = Z(|f|) = Z(f^n)$; $Z(0) = X$; $Z(1) = \emptyset$; and $Z(fg) = Z(f) \cup Z(g)$. Also, if $f \in C(X)$ and $g = |f| \wedge 1$, then $g \in C^*(X)$ and $Z(g) = Z(f)$. Hence, $C(X)$ and $C^*(X)$ yield the same zero-sets. That every zero-set is a G_δ, i. e., a countable intersection of open sets, follows from the fact that $Z(f) = \bigcap_{n=1}^{\infty}\{x \in X : |f(x)| < \frac{1}{n}\}$.

Another concept useful in the study of rings of continuous functions is that of completely separated sets.

Definition 2.2: Two nonempty subsets A and B of X are said to be completely separated (from one another) in

X if there exists a function f in $C^*(X)$ such that $f(A) = \{0\}$, $f(B) = \{1\}$, and $0 \leq f \leq 1$, i. e., $0 \leq f(x) \leq 1$ for all x in X.

Note: Whenever a zero-set Z is a neighborhood of a set A, Z is referred to as a zero-set-neighborhood of A.

Theorem 2.3: Two sets are completely separated if and only if they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set-neighborhoods.

Proof: Let A and B be subsets of X such that $A \subset Z(f)$, $B \subset Z(g)$, and $Z(f) \cap Z(g) = \emptyset$. Note that $|f| + |g|$ has no zeros, and we may define $h: X \rightarrow R$ by $h(x) = \dfrac{|f(x)|}{|f(x)| + |g(x)|}$ for x in X. Hence, $h \in C(X)$, $h(x) = 0$ for $x \in A \subset Z(f)$, and $h(x) = 1$ for $x \in B \subset Z(g)$. Also, $0 \leq h(x) \leq 1$ for all x in X. Hence, A and B are completely separated.

Now, let A and B be completely separated sets in X. Then there exists f in C(X) such that $f(A) = 0$ and $f(B) = 1$. Consider the sets $F = \{x : f(x) \leq \frac{1}{3}\}$ and $G = \{x : f(x) \geq \frac{2}{3}\}$. Now, $F \cap G = \emptyset$, $A \subset \text{int}F$, and $B \subset \text{int}G$. Thus, since $F = \{x : f(x) \leq \frac{1}{3}\} = Z(0 \vee (f - \frac{1}{3}))$ and $G = \{x : f(x) \geq \frac{2}{3}\} = Z(0 \wedge (f - \frac{2}{3}))$, we have that A and B are contained in disjoint zero-set-neighborhoods.

A large portion of the work done in the study of rings of continuous functions is done concerning the idea of extending continuous functions, which brings about the next three definitions.

<u>Definition 2.4</u>: Let S be a subset of X and let f be a function defined on S. Then a function g defined on X is said to be an extension of f to all of X if $g(x) = f(x)$ for $x \in A$. Whenever such a function g exists for f, we say that f is extended to g.

<u>Definition 2.5</u>: A subspace S of X is said to be C-embedded in X if every function in $C(S)$ can be extended to a function in $C(X)$.

<u>Definition 2.6</u>: A subspace S of X is said to be C*-embedded in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$.

<u>Remark 2.7</u>: The fact that a subspace S of X is C*-embedded in X if and only if every function in $C^*(S)$ can be extended to a function in $C(X)$ follows since if a function f in $C^*(S)$ has an extension g in $C(X)$, then f has a bounded extension. That is, if n is a bound for $|f|$, i. e., $-n \leq f(x) \leq n$ for $x \in S$, then $(-n \vee g) \wedge n$ belongs to $C^*(X)$ and agrees with f on S, where n denotes the constant map from X onto n.

The following theorem [2] is an adaptation of Urysohn's theorem that any closed set in a normal space is C*-embedded [8].

<u>Urysohn's Extension Theorem 2.8</u>: A subspace S of X is C*-embedded in X if and only if any two completely separated sets in S are completely separated in X.

The following theorems help to clarify the difference between C*-embedding and C-embedding.

<u>Theorem</u> <u>2.9</u>: A C^*-embedded subset is C-embedded if and only if it is completely separated from every zero-set disjoint from it.

Proof: Let S be a C^*-embedded subset of X. Suppose that S is C-embedded and Z(h) is a zero-set in X disjoint from S. Define $f:S \to R$ by $f(s) = \frac{1}{h(s)}$, for s in S. Then f is a continuous function on S and has an extension g to all of X. Then $gh \in C(X)$; for s in S, $gh(s) = g(s)h(s) = f(s)h(s) = \frac{1}{h(s)}h(s) = 1$; and for x in Z(h), $gh(x) = 0$. Hence, S is completely separated from Z(h).

Now, suppose S is C^*-embedded and completely separated from every zero-set disjoint from S, and let f be any function in C(S). Then $\arctan \circ f$ belongs to $C^*(S)$, and so has an extension to a function g in C(X). The set $Z = \{x \text{ in } X : |g(x)| \geq \frac{\pi}{2}\}$ belongs to Z(X), and is disjoint from S. By the hypothesis, there is a function h in C(X) such that $h(S) = 1$ and $h(Z) = 0$, and $|h| \leq 1$. Then for any s in S, $gh(s) = \arctan(f(s))$ and $|(gh)(x)| < \frac{\pi}{2}$ for all x in X. Hence, $\tan \circ gh$ is a real-valued continuous extension of f to all of X, and S is C-embedded.

<u>Theorem</u> <u>2.10</u>: Every C^*-embedded zero-set is C-embedded.

Proof: Let Z(f) be a C^*-embedded zero-set and Z(h) any zero-set disjoint from Z(f). Then, by Theorem 2.3, Z(f) and Z(h) are completely separated. Now, since Z(f) is completely separated from every zero-set disjoint from it, Z(f) is C-embedded by Theorem 2.9.

Theorem 2.11: Let S be a subspace of X. If every zero-set in S is a zero-set in X, then S is C^*-embedded in X.

Proof: Let S be a subset of X such that every zero-set in S is a zero-set in X. By Urysohn's Extension Theorem, to show S is C^*-embedded in X, it will be sufficient to show any two completely separated sets in S are completely separated in X. So let A and B be two completely separated sets in S. Then A and B are contained in disjoint zero-sets in S which are also disjoint zero-sets in X. Thus, A and B are completely separated in X and it now follows that S is C^*-embedded in X.

CHAPTER III

RINGS OF CONTINUOUS FUNCTIONS ON CONNECTED SPACES

Definition 3.1: Two nonempty sets A and B are said to be separated if and only if $A \cap \mathrm{cl}B = \emptyset = \mathrm{cl}A \cap B$, where $\mathrm{cl}A$ and $\mathrm{cl}B$ denote the closures of A and B, respectively.

Definition 3.2: A subset S is said to be connected if and only if it is not the union of two separated sets.

When studying the connectedness of a topological space X, much can be discovered by considering the square roots of the positive units of C(X), where a unit is an element that has a multiplicative inverse. While C(X) may have many positive units, in looking at the square roots of a positive unit, we may as well look only at the square roots of 1, as is shown by the following theorem.

Theorem 3.3: In C(X) (or C*(X)), all positive units have the same number of square roots.

Proof: We want to consider two positive units in C(X), say u and v. Without loss of generality, we may as well assume $v = 1$. Now suppose that u has m square roots and 1 has n square roots. Let $\{f_1, \ldots, f_n\}$ be the square roots of 1 and let $\{g_1, \ldots, g_m\}$ be the square roots of u.

Suppose $m < n$. Then for $1 \leq i \leq n$, $f_i g_1$ is a square root of u, since $(f_i g_1)^2 = 1g_1^2 = 1u = u$. Suppose for some

i and j we have $f_i g_1 = f_j g_1$. This implies that $f_i g_1 g_1^{-1} = f_j g_1 g_1^{-1}$ or that $f_i = f_j$ and hence for each $i = 1, \ldots, n$, $f_i g_1$ is a distinct square root of u. But this implies that u has n square roots, which is a contradiction.

Suppose $n < m$. Now, $(f_1 g_1)^2 = u$, so $f_1 g_1$ is a square root of u, say $f_1 g_1 = g_1'$. Hence, $f_1 = g_1^{-1} g_1'$. Similarly, $f_2 = g_2^{-1} g_2', \ldots, f_n = g_n^{-1} g_n'$. Now, $g_1^2 = u$ and $(g_1^{-1})^2 g_i^2 = 1$, since $(g_1^{-1} g_i)^2 = (g_1^{-1})^2 g_i^2 = u^{-1} u = 1$ for $i = 1, \ldots, m$. But this implies that 1 has m square roots as long as each of the $g_1^{-1} g_i'$ are distinct for $i = 1, \ldots, m$. But $g_1^{-1} g_i = g_1^{-1} g_j$ implies $g_i = g_j$ and $i = j$. So we again have a contradiction, $n = m$, and all positive units have the same number of square roots.

Thus, with this in mind, we turn our attention to the following characterization of connectedness in terms of the square roots of a positive unit.

<u>Theorem 3.4</u>: A space X is connected if and only if 1 has exactly two square roots.

Proof: Suppose X is connected. Then $f, g \in C(X)$ defined by $f(x) = 1$ and $g(x) = -1$ for all x in X are two square roots of 1. Suppose there exists h in $C(X)$ such that $h(x)h(x) = 1$ and $f(x) \neq h(x) \neq g(x)$ for all x in X. Then $h(x) = 1$ for some x and $h(x) = -1$ for some x. Now consider $A = h^{-1}(1)$ and $B = h^{-1}(-1)$. Clearly, $A \cap B = \emptyset$ and $A \cup B = X$. Suppose there exists an x in A such that x is in the closure of B. Then x must be a limit point of B.

Let U be a neighborhood about $h(x) = 1$ such that -1 is not in U. h is continuous, so there exists an open set V about x such that $h(V) \subset U$. Since x is a limit point of B, there exist y in $V \cap B$. Then $h(y) \in U$ and $h(y) = -1$, a contradiction since $-1 \notin U$. Hence, A does not intersect the closure of B. Similarly, B does not intersect the closure of A and we have that X is the union of two separated sets contradicting that X is connected. Hence, such a map h does not exist and 1 has exactly two square roots.

Now suppose 1 has exactly two square roots, 1 and -1, and suppose X is not connected. Then $X = A \cup B$ where A and B are separated. Define $f: X \to R$ by $f(x) = 1$ if $x \in A$ and $f(x) = -1$ if $x \in B$. Then f is a square root of 1 different from 1 and -1, contradicting that 1 has exactly two square roots. Hence, X is connected.

<u>Definition 3.5</u>: A subset C of a space X is called a component of X if and only if C is a connected set which is not a subset of another connected set in X.

We may also discover the number of components of a space X by counting the number of square roots a positive unit has in $C(X)$.

<u>Theorem 3.6</u>: For finite n, X has n components if and only if 1 has 2^n square roots.

Proof: We first show that if X has n components, then 1 has 2^n square roots by induction. If $n = 1$, X is connected and, by Theorem 3.4, 1 has $2 = 2^1$ square roots. Assume that for a space with k components, 1 has 2^k square roots. Now suppose X has $k + 1$ components denoted by $A_1, \ldots, A_k, A_{k+1}$. Now, the subspace $\bigcup_{i=1}^{k} A_i$ has 2^k square roots of 1, so denote them by f_i for $i = 1, \ldots, 2^k$; and the subspace A_{k+1} has two square roots of 1, say h and g. For each $i = 1, \ldots, 2^k$, define $h_i : X \longrightarrow R$ by $h_i(x) = f_i(x)$ if $x \in \bigcup_{i=1}^{k} A_i$ and $h_i(x) = h(x)$ if $x \in A_{k+1}$. Similarly, for each $i = 1, \ldots, 2^k$, define $g_i : X \longrightarrow R$ by $g_i(x) = f_i(x)$ if $x \in \bigcup_{i=1}^{k} A_i$ and $g_i(x) = g(x)$ if $x \in A_{k+1}$. Thus, for 1 in C(X), we have $2^k + 2^k = 2^{k+1}$ square roots of 1. Hence, for any positive integer n, if X has n components, 1 has 2^n square roots.

Now, suppose 1 has 2^n square roots. If $n = 1$, 1 has 2 square roots and by Theorem 3.4, X is connected and has only one component. Now assume that if 1 has 2^k square roots, X has k components. Thus, if 1 has 2^{k+1} square roots, X has at least $k + 1$ components. If X has $k + 2$ components, then 1 would have 2^{k+2} square roots. Hence, X has $k + 1$ components, and the induction is complete.

We also have another characterization of connectedness in terms of idempotents in C(X) (x is an idempotent if and only if $x^2 = x$).

Theorem 3.7: X is connected if and only if 0 and 1 are the only idempotents in C(X).

Proof: Suppose X is connected and there exists an idempotent f in C(X) such that $0 \neq f \neq 1$. The only idempotents in R are 0 and 1 so for some x, y in X, $f(x) = 0$ and $f(y) = 1$. Thus, by an arguement similar to that for

Theorem 3.4, we have that $A = f^{-1}(0)$ and $B = f^{-1}(1)$ are separated in X and $X = A \cup B$ contradicting that X is connected. Hence, 0 and 1 are the only idempotents in C(X).

Now suppose 0 and 1 are the only idempotents in C(X) and X is not connected. Then $X = A \cup B$ where A and B are separated. Define $f: X \longrightarrow R$ by $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \in B$. Then $f \in C(X)$ and f is an idempotent different from 0 and 1, a contradiction. Hence, X is connected.

Note: As noted in [4], X is the direct sum of A and B, written $X = A \oplus B$, if and only if $X = A + B$ and $A \cap B = 0$.

Theorem 3.8: X is connected if and only if C(X) is not a direct sum of any two rings (except trivially).

Proof: Let X be a connected space and suppose $C(X) = R \oplus S$ where $R \neq \{0\} \neq S$ and R and S are rings. By our initial assumption that all rings discussed will have a unity element, there exist u in R, v in S such that for any r in R, s in S, $ru = r$ and $sv = s$. Thus $u \neq 0 \neq v$, and since $R \cap S = \{0\}$, $u \neq v$. Thus, either u or v is not 1. Hence, C(X) has at least three idempotents, namely 0, u, and v. But by Theorem 3.7, 0 and 1 are the only idempotents in C(X), a contradiction. Hence, C(X) is not a direct sum of any two nontrivial rings.

Now suppose C(X) is not a direct sum of any two nontrivial rings and suppose X is not connected. Then $X = A \cup B$ where A and B are separated sets. Let $R = \{f : f \in C(X), f(A) = 0\}$ and $S = \{f : f \in C(X), f(B) = 0\}$.

To see that $C(X) = R \oplus S$ we first need to verify that R and S are rings. We will do this for R and then a similar arguement will give the same results for S. First note that associativity for R follows from associativity of $C(X)$ and that $0 \in R$. Also, for f and g in R and x in A, $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$; $(fg)(x) = f(x)g(x) = 0 \cdot 0 = 0$; and $-f(x) = -(f(x)) = -0 = 0$; so $f + g$, fg, and $-f$ are in R. Thus R, and hence S, are rings.

Clearly, $\{0\} \subset R \cap S$. Suppose there exists g in $R \cap S$ such that $g \neq 0$. Then there is some x in X such that $g(x) \neq 0$. Now $x \in A$ or $x \in B$, so $g \notin R$ or $g \notin S$ and hence $g \notin R \cap S$. Thus, $R \cap S = \{0\}$.

Now, X is not connected so there exists h in $C(X)$ such that $h(X) = \{0, 1\}$, i. e., $h(A) = \{0\}$ and $h(B) = \{1\}$. So we have that A and B are completely separated and hence contained in disjoint zero-sets. But since $X = A \cup B$ and $A \cap B = \emptyset$, A and B must be zero-sets, say $A = Z(f)$ and $B = Z(g)$. We want to show $C(X) = R + S$, so let $h \in C(X)$ and suppose $h \neq 0$. Now, if $x \in A$, $\frac{f(x)h(x)}{f(x) + g(x)} + \frac{g(x)h(x)}{f(x) + g(x)} = 0 + \frac{g(x)h(x)}{g(x)} = h(x)$ and if $x \in B$, $\frac{f(x)h(x)}{f(x) + g(x)} + \frac{g(x)h(x)}{f(x) + g(x)} = \frac{f(x)h(x)}{f(x)} + 0 = h(x)$. Thus, $h = \frac{fh}{f + g} + \frac{gh}{f + g} \in R + S$. Hence, $C(X) = R \oplus S$, a contradiction. Thus, X is connected

CHAPTER IV

RINGS OF CONTINUOUS FUNCTIONS ON COMPLETELY REGULAR SPACES

Up to this point, most of the remarks made were made pertaining to general topological spaces with no separation axioms assumed. However, at this point it is desirable to stipulate a single class of topological spaces, preferably a class wide enough to include all the interesting spaces, and, at the same time, restrictive enough to allow a significant study of the corresponding rings of continuous functions. The class of topological spaces best suited for these purposes is the class of completely regular spaces. This class of spaces will not restrict us as we shall see by Theorem 4.6.

Definition 4.1: A space X is said to be completely regular provided that it is a Hausdorff space such that, whenever F is a closed set and x is a point in its complement, there exists a function f in $C(X)$ such that $f(x) = 1$ and $f(F) \subset \{0\}$, i. e., F and $\{x\}$ are completely separated.

Several important consequences are obtained from the choice of completely regular spaces. One is that every subspace of a completely regular space is completely regular. Another is that every metric space is completely regular and, in particular, R and all its subspaces are completely regular.

The following is an attempt to demonstrate that a wider class than that of the completely regular spaces need not be considered. The most important theorem to consider here is the one that states for every topological space X, there exists a completely regular space Y such that C(X) is isomorphic to C(Y). However, the following theorems contain important results in themselves, as well as acting as lemmas for the theorem mentioned above. The first two deal specifically with completely regular spaces and the third contains a characterization of a continuous mapping between two topological spaces.

Definition 4.2: Let X be any set and C' an arbitrary subfamily of R^X. The weak topology induced by C' on X is defined to be the smallest topology on X such that all functions in C' are continuous.

The following theorem [2], offered here without proof, while of significance within itself, is included in this paper primarily as a lemma to Theorem 4.6.

Theorem 4.3: Let X be a topological space. Then

(i) C(X) and C*(X) induce the same weak topology on X, and

(ii) If X is a Hausdorff space, then X is completely regular if and only if the topology on X coincides with the weak topology induced by C and C*.

Theorem 4.4: If X is a Hausdorff space whose topology is determined by some subfamily C' of R^X, then X is completely regular.

Proof: Clearly every function in C' is continuous, i. e., $C' \subset C(X)$. Hence, the weak topology induced by C' is contained in the weak topology induced by C. But the weak topology induced by C is always contained in the given topology on X. Thus, the topologies induced by C and C' coincide and hence, by Theorem 4.3, X is completely regular.

Theorem 4.5: Let C' be a subfamily of $C(X)$ that determines the topology of X. A mapping f from a space S into X is continuous if and only if the composite function $g \circ f$ is in $C(S)$ for every g in C'.

Proof: If f is continuous, then obviously $g \circ f$ is continuous.

So suppose $g \circ f$ is in $C(S)$ for every g in C', and consider what happens to subbasic closed sets in X under f^{-1}. These are given, by hypothesis, as the sets of the form $g^{-1}(F)$, where $g \in C'$, and F is a closed set in R. Now, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ which is closed in S since $g \circ f$ is continuous. Thus, f is continuous.

Then, finally, we have the following theorem which elliminates any reason for considering rings of continuous functions on any spaces other than completely regular spaces.

<u>Theorem 4.6</u>: For every topological space X, there exists a completely regular space Y and a continuous map h of X onto Y, such that the mapping T from $C(Y)$ onto $C(X)$ defined by $T(g) = g \circ h$ is an isomorphism.

Proof: Define $x \equiv x'$ in X to mean that $f(x) = f(x')$ for every f in $C(X)$. Evidently, this is an equivalence relation. Let Y be the set of all equivalence classes and define a mapping h of X onto Y by $h(x)$ is the equivalence class that contains x.

With each f in $C(X)$, associate a function g in R^Y such that $g(y)$ is the common value of $f(x)$ at every point x in y. Thus, $f = g \circ h$. Let C' denote the family of all such functions g, i. e., g is in C' if and only if $g \circ h$ is in $C(X)$. Now, let Y have the weak topology induced by C'. Thus, every function in C' is continuous on Y and $C' \subset C(Y)$. The continuity of h now follows from Theorem 4.5. Note that for any function h' in $C(Y)$, $h' \circ h$ is continuous and $h' \in C'$. Thus, $C(Y) \subset C'$ and, hence, $C(Y) = C'$. Also, if y and y' are distinct points of Y, there exists g in C' such that $g(y) \neq g(y')$. Thus, Y is a Hausdorff space and completely regular by Theorem 4.4.

Finally, define T mapping $C(Y)$ onto $C(X)$ by, for g in $C(Y)$, $T(g) = g \circ h$, and let f and g be any two functions in $C(Y)$. Then we have that $T(f + g) = (f + g) \circ h = f \circ h + g \circ h = T(f) + T(g)$ and $T(fg) = (fg) \circ h = (f \circ h)(g \circ h) = T(f)T(g)$. Thus, T is a homomorphism. Also, T is onto and if $f \neq g$, $T(f) = f \circ h \neq g \circ h = T(g)$ and T is 1-1. It now follows that T is an isomorphism.

Remark 4.7: Since $T(f \vee g) = (f \vee g) \circ h = (f \circ h) \vee (g \circ h) = T(f) \vee T(g)$ and $T(f \wedge g) = (f \wedge g) \circ h = (f \circ h) \wedge (g \circ h) = T(f) \wedge T(g)$, we have that T is a lattice isomorphism as well, and that it carries $C^*(Y)$ onto $C^*(X)$. Hence, algebraic or lattice properties that hold for all $C(X)$ and $C^*(X)$ with X completely regular, hold just as well for all $C(X)$ and $C^*(X)$ with X arbitrary.

Theorem 4.8: In a completely regular space, any two disjoint closed sets, one of which is compact, are completely separated.

Proof: Let A and B be disjoint closed subsets of a completely regular space X such that A is compact. Now, for each x in A, $\{x\}$ and B are completely separated and hence contained in disjoint zero-set-neighborhoods, say $x \in Z_x$ and $B \subset Z'_x$. Then $\{Z_x : x \in A\}$ is an open cover for A and thus has a finite subcover, say $\{Z_1, \ldots, Z_n\}$. Note that for each Z_i, $i = 1, \ldots, n$, the corresponding Z'_i contains B. Hence, $\bigcup_{i=1}^{n} Z_i$ and $\bigcap_{i=1}^{n} Z'_i$ are disjoint zero-sets containing A and B, respectively, and A and B are completely separated.

Lemma 4.9: Two sets are completely separated if and only if their closures are completely separated.

Proof: Let A and B be subsets of a space X such that the closure of A and the closure of B, denoted by clA and clB, respectively, are completely separated. Then there exists f in $C(X)$ such that $f(\text{cl}A) = 0$ and $f(\text{cl}B) = 1$. It then follows that $f(A) = 0$ and $f(B) = 1$ and A and B are completely separated.

Now let A and B be two completely separated subsets of X. Then there exists f in $C(X)$ such that $f(A) = 0$, $f(B) = 1$, and $0 \leq f \leq 1$. We want to show that clA and clB are completely separated, so suppose that they are not. Suppose there exists x in clA such that $f(x) = r > 0$. Then $V = (\frac{r}{2}, \frac{3r}{2})$ is an open set in R about $f(x)$. f is continuous, so $f^{-1}(V)$ is an open set about x which contains no points of A. Hence, $x \notin clA$ and $f(clA) = 0$. By a similar arguement we have $f(clB) = 1$ and clA and clB are completely separated.

Theorem 4.10: Every compact set in a completely regular space is C*-embedded.

Proof: Let S be a compact subspace of a completely regular space X, and let A and B be completely separated sets in S. Now $cl_S A$ and $cl_S B$ are completely separated in S, closed in S, and compact. Thus, by Theorem 4.8, clA and clB, and hence A and B, are completely separated in X. Thus, by Urysohn's Extension Theorem, S is C*-embedded.

As there are several familiar countable sets often studied as topological spaces and subspaces, some interesting results have been obtained and included here concerning countable sets. For these, X is assumed to be a completely regular space.

Lemma 4.11: $Z(X)$ is closed under countable intersection.

Proof: Let $\{Z(f_n)\}_{n=1}^{\infty}$ be a countable collection of zero-sets. For each $f_n \in C(X)$, define g_n by $g_n(x) = |f_n(x)| \wedge 2^{-n}$ for x in X, and let g be defined by $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for x in X. Since $|g_n| \leq 2^{-n}$, the series converges uniformly, and hence g is a continuous function. Then, clearly, $Z(g) = \bigcap_{n=1}^{\infty} Z(g_n) = \bigcap_{n=1}^{\infty} Z(f_n)$.

<u>Theorem 4.12</u>: A countable set disjoint from a closed set F is disjoint from some zero-set containing F.

Proof: Let F be a closed set and A a countable set disjoint from F. A is countable, so list the elements of A, say $A = \{x_1, x_2, \ldots\}$. Now, F and each x_i are completely separated and, hence, contained in disjoint zero-sets, say $F \subset Z_i$, $x_i \in Z_i'$. Then $\bigcap_{i=1}^{\infty} Z_i$ is a zero-set about F disjoint from A.

<u>Theorem 4.13</u>: A C-embedded countable set S is completely separated from every closed set disjoint from S.

Proof: Let S be a C-embedded countable set and let F be any closed set disjoint from S. By Theorem 4.12, S is disjoint from a zero-set Z containing F. By Theorem 2.9, S and Z are completely separated and hence, S and F are completely separated.

<u>Theorem 4.14</u>: Any C-embedded countable set is closed.

Proof: Let S be any C-embedded countable set. Let $x \in X \setminus S$. X is Hausdorff, so $\{x\}$ is a closed set and by Theorem 4.13, S and $\{x\}$ are completely separated. Completely separated sets are contained in disjoint zero-set-neighborhoods, so there exist Z and Z' disjoint zero-set-neighborhoods about S and x, respectively. Thus, Z' is a neighborhood about x completely contained in $X \setminus S$, and $X \setminus S$ is open. Hence, S is closed.

As is well known, every continuous real-valued function defined on a compact set is bounded. This idea is generalized in the following definition.

<u>Definition</u> <u>4.15</u>: A completely regular space X is said to be pseudocompact if $C(X) = C^*(X)$, i. e., every continuous function defined on X is bounded.

The following lemma is found in [1] and is used in the proof of Theorem 4.17.

<u>Lemma</u> <u>4.16</u>: Let X be a topological space, $\{V_a : a \in A\}$ an open covering of X, and $g:X \longrightarrow Y$. If for each a in A, g restricted to V_a is continuous, then g is continuous.

The following theorem contains four conditions equivalent to pseudocompactness, and was formed by the combination of theorems found in [1] and [3].

<u>Theorem</u> <u>4.17</u>: The following five properties of a completely regular space X are equivalent:

(1) X is pseudocompact.

(2) Every function in $C^*(X)$ assumes its greatest lower bound and least upper bound for some point or points in X.

(3) If $f \in C^*(X)$, then $f(X)$ is a compact subset of R.

(4) If $\{V_i\}_{i=1}^{\infty}$ is a sequence of open sets such that $V_{i+1} \subset V_i$ for each i, then $\bigcap_{i=1}^{\infty} \mathrm{cl} V_i \neq \emptyset$.

(5) Each countable open covering for X has a finite subfamily whose closures cover X.

Proof: The result is established by proving the implications $(1)\rightarrow(2)\rightarrow(3)\rightarrow(1)\rightarrow(4)\rightarrow(5)\rightarrow(1)$.

Suppose that (2) does not hold. Then there is a function f in $C^*(X)$ which does not assume both of its bounds. We may as well assume f fails to assume its greatest lower bound and denote it by m. Then $f - m$ is a positive function taking on arbitrarily small values, and the function $\frac{1}{f - m}$ is an unbounded function in $C(X)$. Thus, (1) does not hold and the implication $(1)\rightarrow(2)$ is established.

Now, suppose that (3) does not hold. If $f(X)$ is a bounded, non-compact subset of R, then $f(X)$ must not be closed. Let u be a limit point of the set $f(X)$ that is not in $f(X)$. If u is the greatest lower bound or the least upper bound of $f(X)$, then obviously (2) fails. On the other hand, if there are numbers t and s in $f(X)$ such that $t < u < s$, then the set $f(X)$ is not connected, and the sets $A = f^{-1}\{t : t \in f(X), t > u\}$ and $B = f^{-1}\{t : t \in f(X), t < u\}$ are disjoint, nonempty, open sets in X. Suppose that for every positive real number r, the interval $[u, u + r)$ contains points of $f(X)$. Then let g be a function defined on X by $g(p) = f(p)$ for all p in A, and $g(p) = u + 1$ for p in B. Then clearly $g \in C^*(X)$, g has a greatest lower bound u, and $g(p)$ is greater than u for all p in X. A similar construction may be applied if every interval $(u - r, u]$ contains points of $f(X)$. Hence, condition (2) fails if condition (3) fails, and the implication (2)$\rightarrow$(3) is established.

Suppose X is not pseudocompact and let f be a continuous real-valued unbounded function defined on X. Then $f^2 + 1$ is continuous, unbounded above, and always positive. We then have that $\frac{1}{f^2 + 1}$ is continuous, bounded below by 0, and bounded above by 1 and, hence, is in $C^*(X)$. But since $\frac{1}{f^2 + 1}$ takes on arbitrarily small positive values but never zero, we have that $\frac{1}{f^2 + 1}(X)$ is not compact. Hence, the implication (3)$\rightarrow$(1) is established.

Suppose that (4) does not hold and let $\{V_i\}_{i=1}^{\infty}$ be a sequence of open sets such that $V_{i+1} \subset V_i$ for each positive integer i, and suppose that $\bigcap_{i=1}^{\infty} \mathrm{cl}V_i = \emptyset$. X is completely regular, so for each positive integer n, let g_n be a continuous map such that $0 \leq g_n \leq n$ and $g_n(y_n) = n$ for some y_n in V_n, and $g_n(X \setminus V_n) = 0$. Let $F_1 = X \setminus \mathrm{cl}V_1$, $F_2 = X \setminus \mathrm{cl}V_2$, and, in general, $F_n = X \setminus \mathrm{cl}V_n$. Let $x \in X$. If $x \notin \mathrm{cl}V_1$, $x \in F_1$. If $x \in \mathrm{cl}V_1$, there exists a smallest k such that $x \notin \mathrm{cl}V_k$ and $x \in X \setminus \mathrm{cl}V_k = F_k$, since $\bigcap_{i=1}^{\infty} \mathrm{cl}V_i = \emptyset$. Thus, $\{F_i\}_{i=1}^{\infty}$ is an open covering of X. Define $g: X \longrightarrow R$ by $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for all x in X. Now, g restricted to F_n is equal to $g_1 + \cdots + g_{n-1}$ for each positive integer n, which is a finite sum of continuous functions, and hence continuous. Thus, by Lemma 4.16, g is continuous. Suppose g is bounded by some real number r and let $n > r$.

Then there exists y_n in V_n such that $g_n(y_n) = n > r$. Thus, g is continuous and not bounded and X is not pseudo-compact. Hence, the implication $(1) \to (4)$ is established.

Assume (4) and let $\{U_n : n \in Z^+\}$ be a given countable collection of open sets which cover X. Let $V_1 = X \setminus clU_1$, $V_2 = X \setminus (clU_1 \cup clU_2)$, and, in general, $V_n = X \setminus \bigcup_{i=1}^{n} clU_i$ for each positive integer n. Note that $\{V_i\}_{i=1}^{\infty}$ is a descending sequence of open sets and, since $V_n = X \setminus \bigcup_{i=1}^{n} clU_i = \bigcap_{i=1}^{n}(X \setminus clU_i)$, we have that $clV_n \subset \bigcap_{i=1}^{n} cl(X \setminus clU_i) \subset \bigcap_{i=1}^{n} cl(X \setminus U_i) = \bigcap_{i=1}^{n}(X \setminus U_i)$. Now, if no $V_n = \emptyset$, we have by (4) that $\emptyset \neq \bigcap_{i=1}^{\infty} clV_i \subset \bigcap_{i=1}^{\infty}(X \setminus U_i) = X \setminus \bigcup_{i=1}^{\infty} U_i$ which implies that $X \neq \bigcup_{i=1}^{\infty} U_i$ which implies that $\{U_n : n \in Z^+\}$ is not a covering of X. Thus, for some n, $V_n = \emptyset$ which implies $X \setminus \bigcup_{i=1}^{n} U_i = \emptyset$ which implies that $\{U_i : i = 1, \ldots, n\}$ is a finite subfamily of $\{U_n : n \in Z^+\}$ whose closures cover X.

Now assume (5) and let $f \in C(X)$ and let $U_n = \{x : |f(x)| < n\}$ for $n = 1, 2, \ldots$. Thus, $\{U_n : n = 1, 2, \ldots\}$ is an open covering for X and, by (5), there is a positive integer m such that $X = \bigcup_{i=1}^{m} clU_i$. Thus, for $x \in X$, $x \in clU_k$ for some k such that $1 \leq k \leq m$ and $|f(x)| \leq k \leq m$. Hence, f is bounded and X is pseudo-compact. The theorem is now established.

That the study of rings of bounded continuous functions of a space X may be confined to the case in which X is a compact Hausdorff space results from the following two theorems proved by Stone [7].

<u>Theorem</u> <u>4.18</u>: If X and Y are compact Hausdorff spaces such that C*(X) is algebraically isomorphic to C*(Y), then X is homeomorphic to Y.

<u>Theorem</u> <u>4.19</u>: For every completely regular space X, there exists a unique compact Hausdorff space, commonly denoted as βX, having the properties that $X \subset \beta X$, $\mathrm{cl}X = \beta X$, and C*(X) is algebraically isomorphic to C*(βX).

CHAPTER V

RINGS OF CONTINUOUS FUNCTIONS ON NORMAL SPACES

Normal spaces have some additional properties that are useful in the study of rings of continuous functions, but there is relatively little gained by imposing upon a completely regular space the stronger condition of normality. Clearly, since every normal space is completely regular, any result true for completely regular spaces is true for normal spaces. One result, included in the following theorem, is that in a normal space, every closed set is C^*-embedded. In the absence of normality, this result can be replaced by the fact that in a completely regular space, every compact set is C^*-embedded.

Theorem 5.1: The following are equivalent for any Hausdorff space X:

(1) X is normal.

(2) Any two disjoint closed sets are completely separated.

(3) Every closed set is C^*-embedded.

(4) Every closed set is C-embedded.

Proof: The proof of the theorem will be accomplished by proving the implications $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1)$.

(1)$\rightarrow$(2): Suppose X is normal. Then by Urysohn's lemma [5], for any two disjoint closed subsets A and B of X, there is a continuous function f defined on X such that $f(A) = 0$ and $f(B) = 1$. Hence, any two disjoint closed sets are completely separated.

(2)$\rightarrow$(3): Let S be a closed subset of X. By Urysohn's Extension Theorem, to show that S is C*-embedded we need only to show that any two completely separated sets in S are completely separated in X. So let A and B be any two completely separated sets in S. By Lemma 4.9, $cl_S A$ and $cl_S B$ are completely separated. Also, $cl_S A$ and $cl_S B$ are closed in S, and since S is closed in X, $cl_S A$ and $cl_S B$ are closed in X. Thus, by (2), $cl_S A$ and $cl_S B$ are completely separated in X. It now follows that A and B are completely separated in X and that S is C*-embedded.

(3)$\rightarrow$(4): Let A be a closed subset of X, $f \in C(A)$, and consider the homeomorphism $h: R \longrightarrow (-1, 1)$ defined by $h(x) = \dfrac{x}{1 + |x|}$ for all x in R. The map $h \circ f$ mapping A into $(-1, 1)$ is continuous and bounded and thus has a bounded continuous extension F to all of X. Since h is a homeomorphism, h^{-1} is continuous and we have that $h^{-1} \circ F$ is a continuous extension of f to all of X since for each a in A, $(h^{-1} \circ F)(a) = (h^{-1} \circ h \circ f)(a) = f(a)$. Hence, A is C-embedded.

(4)$\rightarrow$(1): Let A and B be disjoint closed subsets of X. Define $f: A \cup B \longrightarrow R$ by $f(A) = 0$ and $f(B) = 1$. Then f is in $C(A \cup B)$ and by (4), since $A \cup B$ is closed in X, f can be extended to F in $C(X)$. Thus, if U and V are disjoint open sets about 0 and 1, respectively, $F^{-1}(U)$ and $F^{-1}(V)$ are disjoint open sets about A and B, respectively. It now follows that X is normal.

The following was proved by James Dugundji [2], and is included here as a lemma to Theorem 5.3.

Lemma 5.2: Let X be a regular space and let A be any infinite subset of X. Then there exists a family $\{U_n : n \geq 0\}$ of open sets whose closures are pairwise disjoint and such that $A \cap U_n \neq \emptyset$ for each $n \geq 1$.

Theorem 5.3: In a normal space, pseudocompactness is equivalent to countable compactness.

Proof: Let X be a countable compact space and let $f \in C(X)$. Then $\{x : |f(x)| < n\}$ for $n = 1, 2, \ldots,$ is a countable open covering for X, and, hence, has a finite subcovering for X. But this implies that f is bounded and X is pseudocompact.

Now suppose X is a normal pseudocompact space. By Theorem 4.17, X pseudocompact is equivalent to saying each countable open covering for X has a finite subfamily whose closures cover X. So we need to find a countable open covering for X which does not satisfy this condition in order to arrive at a contradiction. Now, if X is not countably compact, there exists a countably infinite subset D of X which has no limit point. Then, by Lemma 5.2, there is a collection $\{U_n : n = 1, 2, \ldots\}$ of open sets whose closures are pairwise disjoint and such that $D \cap U_n \neq \emptyset$ for each n. Choose $y_n \in D \cap U_n$. Now, $E = \{y_n : n = 1, 2, \ldots\}$ is a countable closed set, since if x is a limit point of E, $x \in E$. X is normal, so we can find disjoint open sets W and V such that $E \subset W$ and $X \setminus \bigcup_{n=1}^{\infty} U_n \subset V$. Then $\{V\} \cup \{U_n : n = 1, 2, \ldots\}$ is an open covering of X and, since for $n = 1, 2, \ldots,$ $y_n \in clU_n$ and $y_n \notin clV$, $\{V\} \cup \{U_n : n = 1, 2, \ldots\}$ has no finite subfamily whose closures cover X. Thus, X is not pseudocompact and we have a contradiction.

Earlier, it was stated that every zero-set is a G_δ, and we now arrive at the fact that in a normal space, every closed G_δ is a zero-set.

<u>Theorem</u> <u>5.5</u>: Every closed G_δ in a normal space is a zero-set.

Proof: Let A be a closed G_δ in a normal space X. Then $A = \bigcap_{i=1}^{\infty} U_i$ where each U_i is open in X. Note that for each $i = 1, 2, \ldots,$ A and $X \setminus U_i$ are disjoint closed sets, and, since X is normal, A and $X \setminus U_i$ are completely separated and hence contained in disjoint zero-sets, say $A \subset Z_i$ and $(X \setminus U_i) \subset Z_i'$. Then, since Z(X) is closed under countable intersection, $\bigcap_{i=1}^{\infty} Z_i$ is a zero-set containing A. Suppose there exists an x in $\bigcap_{i=1}^{\infty} Z_i$ such that x is not in A. Then $x \in X \setminus A$ implies that $x \in X \setminus \bigcap_{i=1}^{\infty} U_i$ which implies that $x \in \bigcup_{i=1}^{\infty} (X \setminus U_i)$. But this implies there exists an n such that $x \in X \setminus U_n$ and hence $x \in Z_n'$. But $x \in \bigcap_{i=1}^{\infty} Z_i$ implies that $x \in Z_n$ and $Z_n \cap Z_n' = \emptyset$ which is a contradiction. Thus, $x \in A$ and $A = \bigcap_{i=1}^{\infty} Z_i$, a zero-set.

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